

Random unitaries, amenable linear groups and Jordan's theorem

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Abstract

It is well known that a dense subgroup G of the complex unitary group $U(d)$ cannot be amenable as a discrete group when $d > 1$. When d is large enough we give quantitative versions of this phenomenon in connection with certain estimates of random Fourier series on the compact group \bar{G} that is the closure of G . Roughly, we show that if \bar{G} covers a large enough part of $U(d)$ in the sense of metric entropy then G cannot be amenable. The results are all based on a version of a classical theorem of Jordan that says that if G is finite, or amenable as a discrete group, then G contains an Abelian subgroup with index $e^{o(d^2)}$.

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Let G be a compact group. We denote by m_G the normalized Haar measure on G , and by \widehat{G} a maximal family of mutually distinct (up to unitary equivalence) irreducible unitary representations on G . For any $\pi \in \widehat{G}$ let $\chi_\pi(x) = \text{tr}(\pi(x))$ denote as usual its character, so that $\{\chi_\pi \mid \pi \in \widehat{G}\}$ is an orthonormal system in $L_2(G)$.

Let M_d be the space of matrices of size $d \times d$ with complex entries. We use the standard notation $|a| := \sqrt{a^*a}$, i.e. the unique non-negative self-adjoint matrix whose square is a^*a .

Let \mathbb{T} be the unit circle $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$ in the complex plane. Let $U(d) \subset M_d$ denote the group of all unitary matrices.

Our investigation is motivated by the following from [27] (see §5 below):

Theorem 0.1 (Characterization of Subgaussian characters). *Let (G_n) be a sequence of compact groups, let $\pi_n \in \widehat{G_n}$ be nontrivial and let $\chi_n = \chi_{\pi_n}$ as well as $d_n = d_{\pi_n}$. The following are equivalent:*

(i) *There is a constant C such that*

$$\forall n \forall a \in M_{d_n} \quad \text{tr}|a| \leq C \sup_{g \in G_n} |\text{tr}(a\pi_n(g))|.$$

(ii) *There is $C > 0$ such that*

$$\forall n \quad \int \exp(|\chi_n/C|^2) dm_{G_n} \leq \exp(1) (= e).$$

(iii) *There is a constant C such that*

$$\forall n \quad d_n \leq C \int_{U(d_n)} \sup_{g \in G_n} |\text{tr}(u\pi_n(g))| m_{U(d_n)}(du).$$

The property (i) means that the singletons $\{\pi_n\} \subset \widehat{G_n}$ are Sidon with constant C in the sense defined e.g. in [21], while (ii) means that they are central Λ Sidon with a fixed constant in the sense of [22]. See §6 for more background on this. Equivalently, (ii) says that the tail behaviour of χ_n is dominated (uniformly over n) by that of a standard Gaussian normal random variable. In other words the χ_n 's are uniformly subgaussian. Using the Taylor expansion of $x \mapsto \exp x^2$ and Stirling's formula, it is easy to check that (ii) is equivalent to: There is a constant C such that

$$\sup_{p \in 2\mathbb{N}} \|\chi_n\|_p / \sqrt{p} \leq C.$$

See §3 for more on this.

What is a bit surprising in the preceding statement is that the subgaussian integrability property of the character expressed by (ii) implies a rather strong property of the *whole range* of π , that is perhaps better described as a “density” property like in the next corollary.

Corollary 0.2. *The preceding properties are equivalent to*

(iv) *There is a number $0 \leq \alpha < \sqrt{2}$ such that for any n and any $u \in U(d_n)$ there is $t \in G_n$ and $z \in \mathbb{T}$ such that*

$$\text{tr}|u - z\pi_n(t)|^2 \leq \alpha^2 d_n.$$

Proof. Assume (i). For simplicity let $G = G_n$, $d = d_n$ and $\pi = \pi_n$. Then for any $u \in U(d)$ we have $1/C \leq \sup_{g \in G, z \in \mathbb{T}} \Re(zd^{-1}\text{tr}(u\pi(g)))$. Equivalently

$$\inf_{g \in G, z \in \mathbb{T}} d^{-1} \text{tr}|u - z\pi(g)|^2 \leq 2(1 - 1/C^2),$$

and hence (iv) holds.

Conversely assume (iv). Then for any $u \in U(d)$ we have $\inf_{g \in G, z \in \mathbb{T}} d^{-1} \text{tr}|u - z\pi(g)|^2 \leq \alpha^2$, and hence $1 - \alpha^2/2 \leq \sup_{g \in G, z \in \mathbb{T}} \Re(zd^{-1} \text{tr}(u\pi(g))) = \sup_{g \in G} d^{-1} |\text{tr}(u\pi(g))|$. Thus

$$d(1 - \alpha^2/2) \leq \inf_{u \in U(d)} \sup_{g \in G} |\text{tr}(u\pi(g))|.$$

A fortiori (iii) holds. □

Remark 0.3. Note that for any $u, v \in U(d_n)$ there is $z \in \mathbb{T}$ or even $z \in \{-1, 1\}$ such that $\text{tr}|u - zv|^2 \leq 2d_n$. Indeed, the average over all such z 's is equal to $2d_n$.

Remark 0.4. [On irreducibility] If a unitary representation π_n satisfies the inequality appearing in Theorem 0.1 (i), then it is irreducible. Indeed, if P_1, P_2 are mutually orthogonal projections onto invariant subspaces for π_n , and if a is a matrix such that $P_1 a P_2 = a$ we have $\text{tr}(a\pi_n(t)) = 0$ for all t , and hence the inequality in (i) implies $a = 0$, so we must have either $P_1 = 0$ or $P_2 = 0$.

The fundamental example for Corollary 0.2 is very simple: just take $G = \prod U(d_n)$ and let π_n be the coordinates on G . In that case, (iv) obviously holds with $\alpha = 0$.

Until recently, the second author believed naively that the preceding Theorem 0.1 could be applied to finite groups. To his surprise, the first author showed him that it is not so (and he showed him Turing's paper [40] that already invoked Jordan's theorem to emphasize that general phenomenon, back in 1938 !). The reason is roughly that any "large" *finite* subgroup $G \subset U(d)$ contains a "large" Abelian subgroup $\Gamma \subset G$ (and even a normal one), with an upper bound for the index, namely $[G : \Gamma] \leq \exp o(d^2)$ that contradicts the density expressed in (iv), except for the trivial case when d_n stays bounded. More precisely, the root for this lies in a Theorem of Camille Jordan from 1878:

Theorem 0.5. *Any finite subgroup of $U(d)$ has a normal Abelian subgroup of index bounded by a function $f(d)$ depending only on d .*

We will show in Theorem 5.7 that the bound $f(d) \leq (d+1)! = \exp O(d \log(d))$ (see below) implies for any representation $\pi : G \rightarrow U(d)$ with *finite* or *amenable* range

$$\int_{U(d)} \sup_{g \in G} |\text{tr}(u\pi(g))| m_{U(d)}(du) = O((d \log(d))^{1/2}),$$

and

$$\int \exp(|\chi_\pi/C|^2) dm_G \leq e \Rightarrow (1/C) = O((\log d/d)^{1/2})$$

and these orders of growth are sharp.

Thus we cannot have a sequence of finite groups G_n satisfying the properties in Theorem 0.1 or Corollary 0.2 unless the dimensions d_n stay bounded.

Similar questions have been considered previously in the theory of Sidon sets in duals of non-commutative compact groups. We describe this connection in §6. When the representations π_n are defined on a single compact group G (so that $G_n = G$ for all n), in many cases it is known that the dimensions d_n must be bounded. This was proved by Cecchini [9] for G a Lie group and by Hutchinson [22] for G a profinite group. Hutchinson's paper implies the impossibility to have finite groups in Theorem 0.1 with unbounded d_n 's. We should mention that the latter reference (recently pointed out to the second author by A. Figà-Talamanca) already used Jordan's theorem, much like we do.

Although Jordan gave no estimate for the growth of f , it was later proved by Blichfeldt, based on contributions notably by Bieberbach and Frobenius (see Remark 5.11 for details) that this holds with $f(d) = O(d^{c(d/\log d)^2})$ and a fortiori with $f(d) = \exp o(d^2)$. The latter estimate is enough to show that Theorem 0.1 is void for finite groups (see Corollary 5.10 for a precise statement).

More precisely, if $d \geq 71$, any finite group $\Gamma \subset U(d)$ has a normal Abelian subgroup of index at most $(d+1)!$, which is sharp. This more recent bound $(d+1)!$ is due to Collins [10], *but uses the classification of finite simple groups*. The fact that $(d+1)!$ is sharp is easy: just consider the standard irreducible representation on \mathbb{C}^{d+1} of the group of permutations of order $d+1$, restricted to the d -dimensional subspace $(1, \dots, 1)^\perp$, and note that the trivial subgroup is the only normal Abelian subgroup and that its index is $(d+1)!$. Before Collins, a slightly weaker bound had been obtained by Boris Weisfeiler [42] (see [10] for details), before he disappeared in Chile, presumably murdered in early 1985.

1. On Jordan's theorem for amenable subgroups of $U(d)$

If one interprets Corollary 0.2 as a quantitative density property, it is natural to wonder about other properties of dense subgroups of $U(d)$. In particular, since it is well known that for $d \geq 2$ dense subgroups of $U(d)$ cannot be amenable, one may ask whether a group satisfying (iv) (with $\alpha < \sqrt{2}$ fixed and d large enough) must be nonamenable (and a fortiori infinite!). Indeed, this turns out to be true because Theorem 0.5 extends to amenable subgroups of $U(d)$. The proof is a reduction to the finite case, showing that any bound valid for finite subgroups of $U(d)$ will also be true for arbitrary amenable subgroups of $U(d)$. This is due to the first author:

Proposition 1.1. *Let $f(d)$ be a bound in Jordan's Theorem as above. Any subgroup $G \subset U(d)$ that is amenable as a discrete group has a normal Abelian subgroup of index at most $f(d)$. (In particular if $d \geq 71$ this holds with $f(d) = (d+1)!$ by [10]).*

Remark 1.2. Every Abelian subgroup of $U(d)$ can be simultaneously conjugated inside the subgroup D_d of diagonal matrices, so this implies that up to conjugating G by a matrix in $U(d)$ we have $[G : G \cap D_d] \leq (d+1)!$.

Proof of Proposition 1.1. Being amenable G has a solvable subgroup of finite index, by the Tits alternative [39]. The closure of G in the usual topology of $U(d)$ is a compact Lie subgroup with a solvable subgroup of finite index. Without loss of generality we may assume that G is closed. Then the connected component of the identity G^0 is solvable. But solvable compact connected Lie groups are Abelian, isomorphic to $(\mathbb{R}/\mathbb{Z})^k$ for some integer k . By a well-known fact due Borel-Serre [2, Lemme 5.11] and Platonov [41, 10.10], there is a finite subgroup H of G such that $HG^0 = G$. For an integer n , let $T_n := \{t \in G^0; t^n = 1\}$. This is a characteristic subgroup of G^0 , which is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^k$. Hence it is normalized by H and thus $H_n := T_n H$ is a finite subgroup of G . We may apply Jordan's lemma with Collins' bound [10] to this H_n and obtain an Abelian normal subgroup $A_n \leq H_n$ such that $[H_n : A_n] \leq f(d)$. In particular $[T_n : T_n \cap A_n] \leq f(d)$. If $n = p^m$ is a power of a prime p , then T_n has no proper subgroup of index $< p$. So if $p > f(d)$, then A_n contains T_n . Fix such a p . Note that the (increasing) union of all T_{p^m} , $m \geq 1$, is dense in G^0 . This implies that the intersection of all $Z(T_{p^m})$ is $Z(G^0)$, where $Z(K) := \{t \in G; tk = kt \forall k \in K\}$ denotes the centralizer subgroup of K . But any decreasing sequence of compact subgroups of a given compact Lie group is stationary ("Noetherianity"), so this intersection is finite, and hence there is $m \in \mathbb{N}$ such that $Z(T_{p^m}) = Z(G^0)$. It follows that $A := A_{p^m} G^0$ is an Abelian subgroup of G , which is normal and of index

$$[G : A] = [HG^0 : A_{p^m} G^0] = [H_{p^m} G^0 : A_{p^m} G^0] \leq [H_{p^m} : A_{p^m}] \leq f(d).$$

□

2. Consequence for the metric entropy

In this section, we show that Jordan's theorem (or Proposition 1.1) with $f(d) = e^{o(d^2)}$ implies a non trivial property of the metric entropy of any finite (or amenable) subgroup of $U(d)$.

Let (T, δ) be any set T equipped with a metric or pseudo-metric δ . Given a subset $S \subset T$ we denote by $N(S, \delta, \varepsilon)$ the smallest number of a covering of S by open balls of δ -radius ε .

We will mainly consider the distances δ_2 and δ_∞ , corresponding to the Hilbert-Schmidt norm and the operator norm respectively, defined on M_d as follows:

$$\forall u, v \in M_d \quad \delta_2(u, v) = (d^{-1} \text{tr}|u - v|^2)^{1/2}$$

$$\forall u, v \in M_d \quad \delta_\infty(u, v) = \|u - v\|.$$

Note

$$(2.1) \quad \delta_2(u, v) \leq \delta_\infty(u, v).$$

Lemma 2.1. *For any d and any subgroup $G \subset U(d)$ containing an Abelian subgroup of index k we have*

$$\forall \varepsilon \in (0, 2) \quad N(G, \delta_2, \varepsilon) \leq k(2\pi/\varepsilon)^d.$$

Proof. Let $G = \cup_{j \leq k} t_j \Gamma$ be the disjoint decomposition into cosets. Then

$$N(G, \delta_2, \varepsilon) \leq \sum_{j \leq k} N(t_j \Gamma, \delta_2, \varepsilon).$$

Clearly $N(t_j \Gamma, \delta_2, \varepsilon) = N(\Gamma, \delta_2, \varepsilon)$ and since Γ is Abelian the matrices in Γ are simultaneously diagonalizable so that we may assume that Γ is included in the set D_d of all diagonal matrices with entries in \mathbb{T} . Thus by (2.1) we have $N(\Gamma, \delta_2, \varepsilon) \leq N(D_d, \delta_2, \varepsilon) \leq N(D_d, \delta_\infty, \varepsilon) \leq (2\pi/\varepsilon)^d$, from which the lemma follows. □

Remark 2.2. Let $0 < \varepsilon < 2$. Let $A_\varepsilon(d)$ be the smallest number N such that any subgroup $G \subset U(d)$ satisfying

$$N(G, \delta_2, \varepsilon) > N$$

must be non-amenable as a discrete group, and let

$$H_\varepsilon(d) = \log A_\varepsilon(d).$$

By the preceding we have $A_\varepsilon(d) \leq f(d)(2\pi/\varepsilon)^d$, and hence $H_\varepsilon(d) \leq \log f(d) + d \log(2\pi/\varepsilon)$. Thus, assuming $d \geq 71$, the bound in Proposition 1.1 implies a fortiori (by Stirling)

$$H_\varepsilon(d) \leq d \log(d/e) + d \log(2\pi/\varepsilon).$$

We will show in (2.8) that this is asymptotically sharp if we keep $\varepsilon > 0$ fixed and let $d \rightarrow \infty$.

But first we need to clarify the relationship between the various ways to estimate the covering numbers of groups with respect to a translation invariant metric in the presence of a translation invariant probability (Haar) measure.

Let $N'(G, \delta_2, \varepsilon)$ be the smallest number of a covering of G by open balls of δ_2 -radius ε with *centers in G* . It is easy to check that $N(G, \delta_2, \varepsilon) \leq N'(G, \delta_2, \varepsilon) \leq N(G, \delta_2, \varepsilon/2)$ for any $\varepsilon > 0$.

We may consider the closure $\bar{G} \subset U(d)$ of G equipped with its normalized Haar measure $m_{\bar{G}}$. Then by translation invariance, we have

$$(2.2) \quad 1/m_{\bar{G}}(\{g \in \bar{G} \mid \delta_2(g, 1) < \varepsilon\}) \leq N'(\bar{G}, \delta_2, \varepsilon) \leq 1/m_{\bar{G}}(\{g \in \bar{G} \mid \delta_2(g, 1) < \varepsilon/2\}).$$

Obviously, we have $N'(\bar{G}, \delta_2, \varepsilon_1) \leq N'(G, \delta_2, \varepsilon) \leq N'(\bar{G}, \delta_2, \varepsilon)$ for any $\varepsilon_1 > \varepsilon$.

Thus (say) $m_{\bar{G}}(\{g \in \bar{G} \mid \delta_2(g, 1) < 3\varepsilon\}) < 1/A_\varepsilon(d)$ implies that G is non-amenable.

To be more concrete, if we set, say, $\varepsilon = 1/30$, there is $c' > 0$ such that for all d large enough if

$$\log \frac{1}{m_{\bar{G}}(\{g \in \bar{G} \mid \delta_2(g, 1) < 1/10\})} \geq c'd \log d$$

then G is not amenable. We will now show that this is asymptotically sharp.

Remark 2.3 (A case study). Let $\mathcal{G} \subset U(d)$ (actually $\mathcal{G} \subset O(d)$) be the finite subgroup formed of all the matrices of the form

$$u = \sum_1^d \varepsilon_i e_{i, \sigma(i)}$$

where $(\varepsilon_i) \in \{-1, 1\}^d$ and σ is in the symmetric group $S(d)$. The group \mathcal{G} is isomorphic to the semidirect product $\{-1, 1\}^d \rtimes S(d)$. Then

$$\text{tr}(u) = \sum_{i \in \text{Fix}(\sigma)} \varepsilon_i$$

$$\delta_2(u, 1)^2 = 2(d - \text{tr}(u)) = 2 \sum_{i \in \text{Fix}(\sigma)} (1 - \varepsilon_i) + 2(d - |\text{Fix}(\sigma)|)$$

where $\text{Fix}(\sigma) = \{i \mid \sigma(i) = i\}$. For any $\varepsilon > 0$ we have

$$(2.3) \quad \{u \in \mathcal{G} \mid \delta_2(u, 1) < \varepsilon\} = \{u \in \mathcal{G} \mid \text{tr}(u) > d(1 - \varepsilon^2/2)\}.$$

Let X_j be the number of permutations in $S(d)$ with exactly j fixed points. Then for any $0 \leq k < d$

$$(2.4) \quad m_{\mathcal{G}}(\{u \in \mathcal{G} \mid \text{tr}(u) > k\}) = (d!)^{-1} \sum_{j > k} X_j \mathbb{P}(\{S_j > k\})$$

where $S_j = \varepsilon_1 + \dots + \varepsilon_j$ is the sum of j independent (uniformly distributed) choices of signs, and (2.4) is 0 when $k \geq d$. Thus we have for any $0 \leq k < d$ (note that $X_d = 1$ and $\mathbb{P}(\{S_d > k\}) \geq 2^{-d}$)

$$(d!)^{-1} 2^{-d} \leq m_{\mathcal{G}}(\{u \in \mathcal{G} \mid \text{tr}(u) > k\}) \leq (d!)^{-1} \sum_{j > k} X_j.$$

It is easy to see that $X_j = \binom{d}{j} D(d-j)$ where $D(n)$ denotes the number of derangements of an n -element set, i.e. the number of permutations without fixed point in $S(n)$. It is well known (see e.g. [37, p. 67]) that $D(n)$ is of order $n!/e$ when $n \rightarrow \infty$, and more precisely: for any $n \geq 1$ (note $D(1) = 0$)

$$(2.5) \quad D(n) = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!}\right).$$

This shows $n! \geq D(n) \geq n!/3$ for all $n > 1$. Contenting ourselves (for the moment) with the obvious bound $D(d-j) \leq (d-j)!$ we find

$$(2.6) \quad (d!)^{-1} X_j \leq (j!)^{-1},$$

and hence $(d!)^{-1} \sum_{j>k} X_j \leq e(k+1)!^{-1}$. By Stirling's formula $(d!)^{-1} \sum_{j>k} X_j \leq e(e/(k+1))^{k+1}$. Therefore for any $0 \leq k < d$

$$(2.7) \quad (d!)^{-1} 2^{-d} \leq m_{\mathcal{G}}(\{u \in \mathcal{G} \mid \text{tr}(u) > k\}) \leq e(e/(k+1))^{k+1}.$$

Recalling (2.2) and (2.3) and choosing $k = \lfloor d(1 - \varepsilon^2/2) \rfloor$ we find

$$\log N(\mathcal{G}, \delta_2, \varepsilon/2) \geq \log N'(\mathcal{G}, \delta_2, \varepsilon) \geq (k+1) \log((k+1)) - k - 2,$$

from which we deduce, for any $0 < \varepsilon < \sqrt{2}$

$$(2.8) \quad H_{\varepsilon/2}(d) \geq (1 - \varepsilon^2/2) d \log(d/e) - c_3,$$

where c_3 is a fixed constant independent of d .

Remark 2.4. Similarly, assuming $G \subset U(d)$ amenable, let φ be an invariant mean on G . Since both distance and mean are translation invariant it is easy to check that

$$1/\varphi(\{g \in G \mid \delta_2(g, 1) < \varepsilon\}) \leq N'(G, \delta_2, \varepsilon) \leq 1/\varphi(\{g \in G \mid \delta_2(g, 1) < \varepsilon/2\}).$$

By the preceding reasoning $\varphi(\{g \in \bar{G} \mid \delta_2(g, 1) < 3\varepsilon\}) < 1/A_\varepsilon(d)$ would imply that G is not amenable. Therefore we must have

$$\varphi(\{g \in \bar{G} \mid \delta_2(g, 1) < 3\varepsilon\}) \geq 1/A_\varepsilon(d) \geq (f(d)(2\pi/\varepsilon)^d)^{-1},$$

and hence, say taking $\varepsilon = 1/9$

$$\varphi(\{g \in \bar{G} \mid \delta_2(g, 1) < 1/3\}) \geq (18\pi/d)^d / (d+1).$$

3. Subgaussian variables

To conform with a commonly used notation, we set

$$\forall x \in \mathbb{R}_+ \quad \psi_2(x) = e^{x^2} - 1.$$

Given a measure space (T, m) we denote by $L_{\psi_2}(T, m)$ the (Orlicz) space formed of all the measurable complex valued functions F for which there is $c > 0$ such that $\int \psi_2(|F|/c) dm < \infty$. We denote

$$(3.1) \quad \|F\|_{\psi_2} = \inf\{c > 0 \mid \int \psi_2(|F|/c) dm \leq \psi_2(1)\}.$$

When F is real valued with $\int F dm = 0$ and $m(T) = 1$ it is not hard to show that $\|F\|_{\psi_2}$ is equivalent to the smallest constant C such that

$$\forall t \in \mathbb{R} \quad \int \exp(tF - Ct^2/2) dm \leq 1.$$

Since the equality case for $C = 1$ characterizes the standard Gaussian variables, this explains why we view (ii) as a subgaussian estimate.

Using the Taylor expansion of the exponential function it is easy to show that $\|F\|_{\psi_2}$ is equivalent (with absolute equivalence constants) to $F \mapsto \sup_{p \geq 2} \|F\|_p / \sqrt{p}$. More precisely, we can restrict if

we wish to even integers: there is a constant $\lambda > 0$ such that for any complex valued measurable F we have

$$(3.2) \quad \lambda^{-1} \|F\|_{\psi_2} \leq \sup_{n \in \mathbb{N}_*} \|F\|_{2n} / \sqrt{2n} \leq \lambda \|F\|_{\psi_2}.$$

Consider the case when T is a compact group G with $m = m_G$ and let $F = \chi_\pi$ be the character of some $\pi \in \widehat{G}$. Then, for any $i, j \geq 0$, the unitary representation $\pi^{\otimes i} \otimes \bar{\pi}^{\otimes j}$ admits a decomposition into irreducibles that we may write as:

$$(3.3) \quad \pi^{\otimes i} \otimes \bar{\pi}^{\otimes j} = \bigoplus_{\sigma \in \widehat{G}} m_\pi(i, j; \sigma) \sigma,$$

where the integer $m_\pi(i, j; \sigma)$ is the multiplicity (possibly = 0) of σ in $\pi^{\otimes i} \otimes \bar{\pi}^{\otimes j}$. Taking the L_2 -norm of the trace of both sides of (3.3) we find

$$\int |\chi_\pi|^{2(i+j)} dm = \sum_{\sigma \in \widehat{G}} m_\pi(i, j; \sigma)^2.$$

Therefore, the condition

$$\sup_{n \in \mathbb{N}_*} \|\chi_\pi\|_{2n} / \sqrt{2n} \leq C$$

can be reformulated “arithmetically” as saying that for any $i, j \geq 0$ (or merely for all $i \geq 1$ with $j = 0$)

$$\sum_{\sigma \in \widehat{G}} m_\pi(i, j; \sigma)^2 \leq C^{2(i+j)} (2(i+j))^{i+j}.$$

4. Random Fourier series

We describe in this section the connection of Theorem 0.1 and Corollary 0.2 to Gaussian random Fourier series in the style of [25]. More recent information on general Gaussian random processes can be found in [38].

We denote by g_d a random $d \times d$ -matrix with entries $\{g_d(i, j) \mid 1 \leq i, j \leq d\}$ forming an i.i.d. family of complex valued Gaussian variables with $\mathbb{E}|g_d(i, j)|^2 = 1/d$, on a suitable probability space (Ω, \mathbb{P}) .

Let G a compact group G and let $(a_\sigma)_{\sigma \in \widehat{G}}$ be a family of “Fourier coefficients”, i.e. assuming that σ takes its values in $U(d_\sigma)$ we assume that $a_\sigma \in M_{d_\sigma}$. We also assume that $\sum d_\sigma \text{tr}(|a_\sigma|^2) < \infty$. The associated random Fourier series is the random process $(S_t)_{t \in G}$ defined by

$$(4.1) \quad S_t(\omega) = \sum_{\sigma \in \widehat{G}} d_\sigma \text{tr}(a_\sigma g_{d_\sigma}(\omega) \sigma(t)),$$

where the family of random matrices $(g_{d_\sigma})_{\sigma \in \widehat{G}}$ is an independent one. We associate to it the pseudo-distance defined on G by $\delta_S(s, t) = \|S_s - S_t\|_2$. The main results in [25] show that the Dudley-Fernique entropy condition

$$\int_0^\infty (\log N(G, \delta_S, \varepsilon))^{1/2} d\varepsilon < \infty$$

that was known to characterize the a.s. boundedness of (S_t) is also equivalent to the a.s. boundedness of random Fourier series associated to more general randomizations than the Gaussian one. In particular, the same characterization holds for independent unitary matrices uniformly distributed over $\prod_{\sigma \in \widehat{G}} U(d_\sigma)$ in place of $(g_{d_\sigma})_{\sigma \in \widehat{G}}$. In fact these results do not require the irreducibility of the σ 's, as long as one uses the metric entropy associated to δ_S . If one removes the irreducibility

assumption, even the case of S_t reduced to a single sum $S_t = \text{tr}(ag_{d_\pi}\pi(t))$ with $a \in M_{d_\pi}$ is non trivial, and actually it can be argued (by decomposing π into irreducible components) that this case is equivalent to the one in (4.1). In this paper, we concentrate on the even more special case when a is the identity matrix.

Let G be any group and let $\pi : G \rightarrow U(d)$ be a representation. We will estimate the random variable Z_π defined on (Ω, \mathbb{P}) by

$$Z_\pi(\omega) = \sup_{t \in G} |\text{tr}(g_d(\omega)\pi(t))|.$$

For our considerations, it will be essentially equivalent to replace it by the variable

$$u \mapsto \sup_{t \in G} |\text{tr}(u\pi(t))|,$$

defined when u is chosen uniformly in $U(d)$.

We associate to π the (pseudo-)distance δ^π defined on G by

$$\delta^\pi(s, t) = (d^{-1} \text{tr} |\pi(s) - \pi(t)|^2)^{1/2}.$$

We will repeatedly use the observation that

$$(4.2) \quad \delta^\pi(s, t)^2 = 2(1 - d^{-1} \chi_\pi(s^{-1}t)).$$

Let $\varepsilon > 0$. We denote by $N(\pi, \varepsilon)$ the smallest number of a covering of G by open balls of radius ε for the metric δ^π . We then introduce the so-called metric entropy integral

$$\mathcal{I}(\pi) = \int_0^2 (\log N(\pi, \varepsilon))^{1/2} d\varepsilon.$$

Note $\log N(\pi, \varepsilon) = 0$ for all $\varepsilon > 2$ since the diameter of G is at most 2.

In the present very particular situation the Dudley-Fernique theorem for Gaussian random Fourier series (see [25]), says that there are numerical positive constants b_1, b_2 such that for any G , π and d

$$(4.3) \quad b_1 \mathcal{I}(\pi) \leq \mathbb{E} Z_\pi \leq b_2 \mathcal{I}(\pi).$$

By elementary arguments (based on the translation invariance both of the metric δ^π and the measure m_G) we have (as in (2.2)) for any π

$$(4.4) \quad m_G(\{t \mid \delta^\pi(t, 1) < \varepsilon\})^{-1} \leq N(\pi, \varepsilon) \leq m_G(\{t \mid \delta^\pi(\pi, 1) < \varepsilon/2\})^{-1},$$

so that $\mathcal{I}(\pi)$ is equivalent to $\mathcal{I}'(\pi) = \int_0^2 (-\log m_G(\{t \mid \delta^\pi(t, 1) < \varepsilon\}))^{1/2} d\varepsilon$.

By the comparison arguments from [25] we also have for suitable constants b_1, b_2

$$(4.5) \quad b_1 \mathcal{I}(\pi) \leq \int_{U(d)} \sup_{t \in G} |\text{tr}(u\pi(t))| m_{U(d)}(du) \leq b_2 \mathcal{I}(\pi).$$

A fortiori, this shows that

$M_u = \int_{U(d)} \sup_{t \in G} |\text{tr}(u\pi(t))| m_{U(d)}(du)$ and $M_g = \mathbb{E} \sup_{t \in G} |\text{tr}(g_d\pi(t))|$ ($= \mathbb{E} Z_\pi$) are equivalent.

Actually, in the present situation, the latter equivalence can be proved directly very easily, using the matricial version of the “contraction principle” in [25, p. 82]. We briefly indicate the argument:

one direction uses the fact that the polar decomposition of g_d is such that g_d (with respect to \mathbb{P}) has the same distribution as the variable $u|g_d|$ with respect to $m_{U(d)} \times \mathbb{P}$ on the product $U(d) \times \Omega$. This implies that $u\mathbb{E}[g_d]$ can be obtained from g_d by the action of a conditional expectation. Since $\mathbb{E}[g_d] = b_d I$ with $b_d \geq b$ for some numerical constant $b > 0$, this gives us $bM_u \leq M_g$. To prove the converse, we note (“contraction principle”) that a convex function on M_d is maximized on the unit ball at an extreme point, i.e. at a matrix in $U(d)$, and so for any fixed ω , we have

$$\int_{U(d)} \sup_{t \in G} |\text{tr}(ug_d(\omega)\pi(t))| m_{U(d)}(du) \leq \|g_d(\omega)\| \int_{U(d)} \sup_{t \in G} |\text{tr}(u\pi(t))| m_{U(d)}(du)$$

and hence after integration in ω with respect to \mathbb{P}

$$\int_{U(d)} \sup_{t \in G} |\text{tr}(ug_d(\omega)\pi(t))| m_{U(d)}(du) d\mathbb{P}(\omega) \leq \mathbb{E}\|g_d\| \int_{U(d)} \sup_{t \in G} |\text{tr}(u\pi(t))| m_{U(d)}(du).$$

Since, as is well known, $\mathbb{E}\|g_d\|$ remains bounded by a constant b' when $d \rightarrow \infty$ (see e.g. [25, p. 78]) this implies the converse inequality $M_g \leq b'M_u$.

Since $N(\pi, \epsilon)$ is a non-increasing function of ϵ , note also the elementary minoration

$$(4.6) \quad \sup_{0 < \epsilon \leq 2} \epsilon (\log N(\pi, \epsilon))^{1/2} \leq \mathcal{I}(\pi),$$

which, by (4.5), gives us the lower bound

$$(4.7) \quad b_1 \sup_{0 < \epsilon \leq 2} \epsilon (\log N(\pi, \epsilon))^{1/2} \leq \int_{U(d)} \sup_{t \in G} |\text{tr}(u\pi(t))| m_{U(d)}(du).$$

In the Gaussian case, we also have $b_1 \sup_{0 < \epsilon \leq 2} \epsilon (\log N(\pi, \epsilon))^{1/2} \leq M_g$. The latter is known as the Sudakov minoration (see e.g. [30, p. 69] or [24, p. 80]). While the preceding 2-sided bound (4.3) requires the translation invariance of the distance (or the stationarity of the associated Gaussian process), Sudakov’s lower bound holds for general Gaussian processes.

5. Proofs

We first indicate where the proof of Theorem 0.1 can be found. By [27, Cor. 5.4] (see also [32]) (i) and (ii) in Theorem 0.1 are equivalent. Moreover, by [27, Prop. 5.3] they are equivalent to (iii). Note that complete details for this can be found in [32] (together with a correction to another assertion in [27]). As for the equivalence between (ii) and (iii), a more precise two sided inequality holds for the corresponding best possible constants:

Lemma 5.1. *Let G be a compact group, $d \geq 1$ and $\pi : G \rightarrow U(d)$ an irreducible unitary representation. We set*

$$C_2(\pi) = \|\chi_\pi\|_{\psi_2} \text{ and } C_3(\pi) = \frac{d}{\int_{U(d)} \sup_{t \in G} |\text{tr}(u\pi(t))| m_{U(d)}(du)}.$$

There is a numerical constant $K > 0$ such that for any G, d, π

$$K^{-1}C_3(\pi) \leq C_2(\pi) \leq KC_3(\pi).$$

Proof. Firstly we will show $K^{-1}C_3(\pi) \leq C_2(\pi)$. Note that $1 = (\text{tr}(|u|^2/d))^{1/2} = \|\text{tr}(u\pi)\|_2 \leq \|\text{tr}(u\pi)\|_\infty$ for any fixed $u \in U(d)$, therefore $C_3(\pi) \leq d$. Let $0 < \varepsilon < \sqrt{2}$. Let $C = \|\chi_\pi\|_{\psi_2}$. Then $\|\Re(\chi_\pi)\|_{\psi_2} \leq C$ and hence

$$\exp((d/C)^2(1 - \varepsilon^2/2)^2)m_G(\{\Re(\chi) > d(1 - \varepsilon^2/2)\}) \leq e.$$

Taking the square root of the log, we find

$$(d/C)(1 - \varepsilon^2/2) \leq 1 + (\log 1/m_G(\{\Re(\chi) > d(1 - \varepsilon^2/2)\}))^{1/2}$$

and hence using (4.4)

$$\leq 1 + (\log N(\pi, \varepsilon))^{1/2}.$$

By (4.7) this is

$$\leq 1 + (b_1\varepsilon)^{-1} \int_{U(d)} \sup_{t \in G} |\text{tr}(u\pi(t))| m_{U(d)}(du) \leq 1 + (b_1\varepsilon)^{-1} d/C_3(\pi).$$

Thus we obtain

$$(d/C_2(\pi))(1 - \varepsilon^2/2) \leq 1 + (b_1\varepsilon)^{-1} d/C_3(\pi),$$

or equivalently (multiplying by $C_2(\pi)C_3(\pi)/d$)

$$C_3(\pi)(1 - \varepsilon^2/2) \leq C_2(\pi)C_3(\pi)/d + (b_1\varepsilon)^{-1}C_2(\pi)$$

and since $C_3(\pi) \leq d$, choosing say, $\varepsilon = 1$, we obtain $C_3(\pi) \leq KC_2(\pi)$ with $K = 2(1 + b_1^{-1})$.

We now turn to the converse direction. Let

$$\Phi(a) = \int_{U(d)} \sup_{g \in G} |\text{tr}(ua\pi(g))| m_{U(d)}(du).$$

We first claim that for any matrix $a \in M_d$

$$(5.1) \quad \text{tr}|a| \leq C_3(\pi)\Phi(a).$$

This follows from a simple averaging argument. Indeed let $a = u|a|$ be the polar decomposition ($|a|^2 = a^*a$), then $\Phi(a) = \Phi(|a|)$ so that to show (5.1) it suffices to show that $\text{tr}(a) \leq C_3(\pi)\Phi(a)$ for all a in M_d . Then for any $s \in G$ we have $\Phi(a) = \Phi(\pi(s)a)$ (by translation in variance over $U(d)$) and $\Phi(a) = \Phi(a\pi(s^{-1}))$ (by translation in variance over G). By convexity

$$\Phi(a) = \Phi(\pi(s)a\pi(s^{-1})) \geq \Phi\left(\int \pi(s)a\pi(s^{-1})m(ds)\right) = \Phi(\text{tr}(a)I/d) = \text{tr}(a)\Phi(I)/d$$

and this gives us (5.1), since $\Phi(I) = d/C_3(\pi)$ by definition of $C_3(\pi)$.

We now interpret (5.1) as saying that the norm of a natural inclusion J between two normed spaces is at most $C_3(\pi)$: we write $\|J : X \rightarrow M_d^*\| \leq C_3(\pi)$ (the space X is M_d equipped with the norm Φ). By duality the inequality (5.1) means that

$$(5.2) \quad \|J^* : M_d \rightarrow X^*\| \leq C_3(\pi).$$

By the duality theorem in [25, p. 116] the dual of X can be identified (up to a fixed isomorphic constant) with the space of Fourier multipliers from $L_2(G) \rightarrow L_{\psi_2}(G)$. It follows that for any $b \in M_d$

$$(5.3) \quad \|J^*(b)\|_{X^*} = \sup\{|\text{tr}(ba)| \mid \Phi(a) \leq 1\} \simeq \sup\{\|\text{tr}(ba\pi)\|_{\psi_2} \mid \|\text{tr}(a\pi)\|_2 \leq 1\}.$$

Since (5.2) implies $\|J^*(b)\|_{X^*} \leq C_3(\pi)\|b\|_{M_d}$, taking $b = a = I$ we obtain from (5.3) the announced bound

$$C_2(\pi) = \|\mathrm{tr}(\pi)\|_{\psi_2} \leq KC_3(\pi).$$

□

Remark 5.2. More generally if we work with a subset $\Lambda \subset \widehat{G}$ the duality theorem says that the best constant in

$$(5.4) \quad \forall a_\pi \quad \sum_{\Lambda} d_\pi \mathrm{tr}|a_\pi| \leq C\mathbb{E} \left\| \sum_{\Lambda} d_\pi \mathrm{tr}(u_\pi a_\pi \pi) \right\|_\infty$$

and

$$(5.5) \quad \forall a_\pi \quad \left\| \sum_{\Lambda} d_\pi \mathrm{tr}(a_\pi \pi) \right\|_{\psi_2} \leq C \left\| \sum_{\Lambda} d_\pi \mathrm{tr}(a_\pi \pi) \right\|_2$$

are equivalent.

Moreover by the same averaging argument (based on irreducibility of the π 's) the best constant in (5.4) is the same if we restrict (5.4) to the case when the a_π 's are scalar matrices.

Remark 5.3. Let $C_1(\pi)$ be the best constant associated to (i) in Theorem 0.1. More precisely (this is the Sidon constant of $\{\pi\}$ in the sense of §6), we define

$$C_1(\pi) = \sup\{|\mathrm{tr}|a|| \mid a \in M_d, \sup_{g \in G} |\mathrm{tr}(a\pi(g))| \leq 1\}.$$

Obviously $C_3(\pi) \leq C_1(\pi)$. In the converse direction, the best known estimate seems to be

$$(5.6) \quad C_1(\pi) \leq K'C_3(\pi)^2 \log(1 + C_3(\pi)),$$

for some numerical constant K' . To check this we first invoke again the duality theorem in [25, p. 116]. This implies that for any $a \in M_d$ we have

$$\|\mathrm{tr}(\pi)\|_{\psi_2} \leq K''C_3(\pi)(\mathrm{tr}(|a|^2))^{1/2},$$

for some numerical constant K'' . Then (5.6) can be deduced from the proof of [31, Th. 3.7] if one takes into account the logarithmic growth described [31, Rem. 1.16].

The next statement follows from Theorem 0.1 by the same simple argument already used in the Abelian case in [28].

Corollary 5.4. *The properties in Theorem 0.1 are equivalent to the following ones:*

(v) *There are numbers $\beta > 0$ and $\alpha > 0$ such that for any n there is a subset $A_n \subset G_n$ with $|A_n| \geq e^{-1}e^{\alpha d_n^2}$ such that*

$$\forall s \neq t \in A_n \quad \|\pi_n(s) - \pi_n(t)\| > \beta.$$

(v)' *There are numbers $\beta' > 0$ and $\alpha' > 0$ such that for any n there is a subset $A'_n \subset G_n$ with $|A'_n| \geq e^{-1}e^{\alpha' d_n^2}$ such that*

$$\forall s \neq t \in A'_n \quad (d_n^{-1/2} \mathrm{tr}|\pi_n(s) - \pi_n(t)|^2)^{1/2} > \beta'.$$

Proof. We will first show that (i) and (ii) in Theorem 0.1 are equivalent to (v)'. This is an easy consequence of the subgaussian estimate (ii) and of (4.4). Indeed, let $\pi = \pi_n$, $G = G_n$. Recall $\delta^\pi(t, 1)^2 = 2(1 - d_\pi^{-1} \Re \chi_\pi(t))$. Therefore (ii) implies assuming $\varepsilon < \sqrt{2}$

$$(5.7) \quad m_G(\{t \mid \delta^\pi(t, 1) < \varepsilon\}) = m_G(\{t \mid \Re \chi_\pi(t) > d_\pi(1 - \varepsilon^2/2)\}) \leq ee^{-\gamma d_\pi^2}$$

where $\gamma = \beta(1 - \varepsilon^2/2)^2$. From this follows by (4.4)

$$N(\pi, \varepsilon) \geq m_G(\{t \mid \Re \chi_\pi(t) > d_\pi(1 - \varepsilon^2/2)\})^{-1} \geq e^{-1} e^{\gamma d_\pi^2}.$$

Let $A \subset G$ be a maximal subset of points such that

$$\forall s \neq t \in A \quad \delta^\pi(s, t) = (d^{-1} \text{tr} |\pi(s) - \pi(t)|^2)^{1/2} > \varepsilon/2.$$

Clearly, $|A| \geq N(\pi, \varepsilon)$. Therefore, (v)' follows with $\beta' = \varepsilon/2$ (which can be any number $< 1/\sqrt{2}$), and $\alpha' = \gamma^{1/2}$. This shows (ii) implies (v)'. Conversely, assume (v)'. We will show that (iii) holds. Indeed, (v)' implies a lower bound $N(\pi, \beta'/2) \geq |A_n| \geq e^{-1} e^{\alpha' d_n^2}$, and plugging this into (4.5) and (4.6) we immediately derive (iii).

To complete the proof we will show that (v) and (v)' are equivalent. Clearly (v)' implies (v). For the converse, we will use the following non-commutative analogue of a result from approximation theory (see e.g. [7]).

Sublemma. *Let $B_d = \{x \in M_d \mid (d^{-1/2} \text{tr} |x|^2)^{1/2} \leq 1\}$. For any $\xi > 0$ there is a constant r_ξ such that, for any d , we have*

$$N(B_d, \delta_\infty, r_\xi) \leq \exp(\xi d^2).$$

To prove the sublemma, given subsets K_1, K_2 of M_d let us denote for $\varepsilon > 0$ by $N(K_1, \varepsilon K_2)$ the smallest number of a covering of K_1 by translates of εK_2 . If K_3 is another set, obviously we note for later use that for any $r > 0$

$$(5.8) \quad N(K_1, \varepsilon r K_2) \leq N(K_1, \varepsilon K_3) N(\varepsilon K_3, \varepsilon r K_2) = N(K_1, \varepsilon K_3) N(K_3, r K_2).$$

Let \mathcal{B}_d be the unit ball of M_d (equipped with the operator norm). Note $\mathcal{B}_d \subset B_d$. The sublemma is clearly equivalent to the claim that for any $\xi > 0$ there is r_ξ such that $N(B_d, r_\xi \mathcal{B}_d) \leq \exp(\xi d^2)$. There are many possible proofs of the latter. We choose one for which we have the references at hand. By [30, Cor. 5.12 p. 80] (up to a change of notation) there is an absolute constant C such that

$$\sup_{\varepsilon > 0} \varepsilon (\log N(B_d, \varepsilon \mathcal{B}_d)) \leq C d \mathbb{E} \|g_d\|.$$

Since, as we already mentioned, $\mathbb{E} \|g_d\|$ remains bounded when $d \rightarrow \infty$ (see e.g. [25, p. 78]) we may modify the absolute constant C so that

$$\sup_{\varepsilon > 0} \varepsilon (\log N(B_d, \varepsilon \mathcal{B}_d)) \leq C d.$$

Then, choosing $\varepsilon = C/\xi^{1/2}$, we find as announced $N(B_d, r_\xi \mathcal{B}_d) \leq \exp(\xi d^2)$ with $r_\xi = C/\xi^{1/2}$, completing the proof of the sublemma.

We now show that (v) \Rightarrow (v)'. Assume (v). Again we set $G = G_n, \pi = \pi_n$ and $d = d_\pi$. Then (v) implies $|A_n| \leq N(\pi(G), (\beta/2) \mathcal{B}_d)$. By (5.8) we have for any $r > 0$

$$|A_n| \leq N(\pi(G), (\beta/2r) B_d) N(B_d, r \mathcal{B}_d).$$

Choosing $r = r_\xi$ gives us

$$|A_n| \exp(-(\xi d^2)) \leq N(\pi(G), (\beta/2r_\xi) B_d),$$

then choosing (say) $\xi = \alpha/2$ we obtain $e^{-1} e^{\alpha d^2/2} \leq N(\pi(G), (\beta/2r_\xi) B_d)$. From this considering as usual a maximal set of points $A' \subset G$ such that $(d^{-1/2} \text{tr} |\pi(s) - \pi(t)|^2)^{1/2} > \beta/4r_\xi$ for all $s \neq t \in A'$, we have necessarily $|A'| \geq N(\pi(G), (\beta/2r_\xi) B_d)$. Thus we obtain (v)' with $\beta' = \beta/4r_{\alpha/2}$. \square

Lemma 5.5. *Let G be a group with an Abelian subgroup Γ of index $k < \infty$. Let $\pi : G \rightarrow U(d)$ be a unitary representation. Then*

$$(5.9) \quad \mathbb{E} \sup_{t \in G} |\operatorname{tr}(g_d \pi(t))| \leq \sqrt{2 \log k} + \sqrt{d}.$$

Proof. Up to an extra constant factor, this can be easily derived from Lemma 2.1 and (4.3) by plugging the estimate of Lemma 2.1 into the upper bound of (4.3). We give a direct proof for the convenience of the reader. Let $G = \cup_{j \leq k} t_j \Gamma$ be the disjoint decomposition into cosets. Let

$$Y_j = \sup_{t \in t_j \Gamma} |\operatorname{tr}(g_d \pi(t))| = \sup_{\gamma \in \Gamma} |\operatorname{tr}(g_d \pi(t_j) \pi(\gamma))|.$$

Then

$$\sup_{t \in G} |\operatorname{tr}(g_d \pi(t))| = \sup_j Y_j.$$

Since $\{\pi(\gamma) \mid \gamma \in \Gamma\}$ are commuting unitary matrices, they are simultaneously diagonalizable, i.e. $\exists V \in U(d)$ such that $\pi(\gamma) = V D(\gamma) V^{-1}$ where $D(\gamma)$ is a diagonal matrix. Then $\operatorname{tr}(g_d \pi(t_j) \pi(\gamma)) = \operatorname{tr}(V^{-1} g_d \pi(t_j) V D(\gamma))$. Moreover, $V^{-1} g_d \pi(t_j) V \stackrel{\text{dist}}{=} g_d$. Therefore, $Y_j \stackrel{\text{dist}}{=} W(g_d)$ where

$$W(g_d) = \sup_{\gamma \in \Gamma} |\operatorname{tr}(g_d D(\gamma))| = \sup_{\gamma \in \Gamma} \left| \sum_i g_d(i, i) D_{i,i}(\gamma) \right| \leq \sum_i |g_d(i, i)|.$$

The function $[x_d(i, j)] \mapsto W(x_d/\sqrt{d})$ is Lipschitz on $\mathbb{C}^{d^2} = \mathbb{R}^{2d^2}$ (equipped with the Euclidean norm) with distortion ≤ 1 . Therefore (see [29, p. 181] or [23, 24]) for any $\lambda \in \mathbb{R}$

$$\mathbb{E} \exp \lambda(W - \mathbb{E}W) \leq \exp(\lambda^2/2).$$

Now

$$\mathbb{E} \exp \sup_j \lambda(Y_j - \mathbb{E}Y_j) \leq \sum_j \mathbb{E} \exp \lambda(Y_j - \mathbb{E}Y_j) \leq k \exp(\lambda^2/2).$$

A fortiori by convexity for any $\lambda > 0$

$$\exp(\lambda \mathbb{E} \sup_j (Y_j - \mathbb{E}Y_j)) \leq k \exp \lambda^2/2.$$

Let $R = \mathbb{E} \sup_j (Y_j - \mathbb{E}Y_j)$. We have $\exp \lambda R - \lambda^2/2 \leq k$ and hence (take $\lambda = R$)

$$R \leq \sqrt{2 \log k}.$$

Clearly

$$\mathbb{E} \sup_j Y_j \leq \mathbb{E} \sup_j (Y_j - \mathbb{E}Y_j) + \sup_j \mathbb{E}Y_j \leq R + \mathbb{E} \sum_i |g_d(i, i)| \leq R + d \mathbb{E}|g_d(1, 1)| \leq R + \sqrt{d}.$$

From this the announced result follows. □

Lemma 5.6. *In the situation of Lemma 5.5, we have*

$$\forall \varepsilon < \sqrt{2} \quad m_G(\{\Re \chi_\pi > d(1 - \varepsilon^2/2)\}) \geq k^{-1}(\varepsilon/2\pi)^d,$$

and

$$(5.10) \quad \|\chi_\pi\|_{\psi_2} \geq c' \min\{\sqrt{d}, \frac{d}{\sqrt{\log k}}\},$$

where $c' > 0$ is a numerical constant.

Proof. Going back to the definitions, we find that $N(\pi, \varepsilon)$ is essentially the same as $N(\pi(G), \delta_2, \varepsilon)$, but using only balls centered in $\pi(G)$. Thus $N(\pi, 2\varepsilon) \leq N(\pi(G), \delta_2, \varepsilon) \leq N(\pi, \varepsilon)$. By Lemma 2.1 and (4.4)

$$m_G(\{\Re \chi_\pi > 1 - 2\varepsilon^2\}) \geq k^{-1}(\varepsilon/2\pi)^d.$$

Let $r = \|\chi_\pi\|_{\psi_2}$. Note $\|\Re \chi_\pi\|_{\psi_2} \leq r$. We have for any $s > 0$

$$m\{\Re \chi_\pi > s\} \leq e \exp(-(s^2/r^2)),$$

and hence with $s = d(1 - 2\varepsilon^2)$ we find

$$k^{-1}(\varepsilon/2\pi)^d \leq e \exp(-(d^2(1 - 2\varepsilon^2)^2/r^2).$$

Choose say $\varepsilon = 1/2$. Then a simple calculation leads to the announced lower bound for r . \square

Theorem 5.7. *If G is finite or amenable (as a discrete group) then for any $d > 1$ and any representation $\pi : G \rightarrow U(d)$ we have*

$$(5.11) \quad \int_{U(d)} \sup_{g \in G} |\text{tr}(u\pi(g))| m_{U(d)}(du) \leq c_1 \sqrt{d \log(d)}$$

and

$$(5.12) \quad \|\chi_\pi\|_{\psi_2} \geq c_2 \sqrt{d/\log(d)},$$

where c_1 and c_2 are positive constants independent of d .

Proof. This follows from Proposition 1.1 and (5.9) and (5.10) applied with $k = (d+1)!$. \square

Remark 5.8. The proof of (5.12) is similar to but simpler than the one used by Hutchinson for profinite groups in [22], but since he used a weaker bound for $f(d)$ his estimate is weaker.

Remark 5.9. The estimates (5.11) and (5.12) are asymptotically optimal. This can be seen by considering the same case study $\mathcal{G} \subset U(d)$ as in Remark 2.3. Indeed, let $Z_0 = \sup_{\sigma \in S(d)} |\sum_1^d g_d(i, \sigma(i))|$. We have

$$Z_0 \leq \sup_{v \in \mathcal{G}} |\text{tr}(uv)|.$$

Let $V_\sigma = \sum_1^d g_d(i, \sigma(i))$. Now if $\sigma, \sigma' \in S(d)$ differ on exactly k places we have

$$\|V_\sigma - V_{\sigma'}\|_2^2 = 2k/d.$$

Therefore if $r(\sigma, \sigma')$ denotes the number of i 's where $\sigma(i) \neq \sigma'(i)$ and if u_σ denotes the unitary matrix associated to σ , we have

$$\|V_\sigma - V_{\sigma'}\|_2 = (2r(\sigma, \sigma')/d)^{1/2} = \delta_2(u_\sigma, u_{\sigma'}).$$

Note that $r(\sigma, 1) = d - j$ iff σ has exactly j fixed points.

We claim that the Sudakov minoration implies $\mathbb{E} Z_0 \geq c_3 \sqrt{d \log d}$ for some $c_3 > 0$ independent of d . Indeed, applying (2.2) to the copy of $S(d)$ formed by the subgroup $G_S = \{u_\sigma \mid \sigma \in S(d)\} \subset U(d)$, we find

$$1/m_{S(d)}(\{\sigma \in S(d) \mid (2r(\sigma, 1)/d)^{1/2} < \varepsilon\}) \leq N(G_S, \delta_2, \varepsilon).$$

By (2.6)

$$m_{S(d)}(\{\sigma \in S(d) \mid (2r(\sigma, 1)/d)^{1/2} < \varepsilon\}) = \sum_{j > d(1-\varepsilon^2/2)} X_j/d! \leq \sum_{j > d(1-\varepsilon^2/2)} 1/j!.$$

Note that for any $1 \leq k \leq d$

$$(5.13) \quad \sum_{j>k-1} 1/j! \leq e/k! \leq e(e/k)^k.$$

Taking e.g. $\varepsilon = 1$ and $k - 1 = \lfloor d/2 \rfloor$, the latter sum is $\leq e(2e/d)^{d/2}$ thus

$$e^{-1}(d/2e)^{d/2} \leq N(G_S, \delta_2, 1).$$

Note that

$$\sup_{u \in G_S} |\mathrm{tr}(g_d u)| = \sup_{\sigma} |V_{\sigma}| = Z_0,$$

and hence by (4.3) and (4.6)

$$(b_1/\sqrt{2e})\sqrt{d \log d} \approx b_1 \sqrt{\log(e^{-1}(d/2e)^{d/2})} \leq \mathbb{E}Z_0.$$

This proves our claim and a fortiori that $\mathbb{E} \sup_{v \in \mathcal{G}} |\mathrm{tr}(g_d v)| \geq c_3 \sqrt{d \log d}$. By the equivalence of M_g and M_u observed after (4.5) this proves that (5.11) is optimal.

We now turn to (5.12). For any $u = \sum_1^d \varepsilon_i e_{i, \sigma(i)} \in \mathcal{G}$, let $\chi(u) = \mathrm{tr}(u) = \sum_{i \in \mathrm{Fix}(\sigma)} \varepsilon_i$. Then for any $s > 0$

$$m_{\mathcal{G}}(\{|\chi| > s\}) = 2m_{\mathcal{G}}(\{\chi > s\}) \leq (2/d!) \sum_{j>s} X_j \leq 2 \sum_{j>s} 1/j!.$$

We claim that there are positive constants (independent of d) c_4, c_5 and $c_6 > e$ such that

$$\forall s > c_6 \quad m_{\mathcal{G}}(\{|\chi| > s\}) \leq c_4 \exp(-(c_5 s \log s)).$$

Indeed, this is easy to derive from (5.13). Now since $m_{\mathcal{G}}(\{|\chi| > s\}) = 0$ for all $s > d$ we may restrict consideration to $e < s < d$ for which $s \log s = s^2(\log s/s) \geq s^2(\log d/d)$, and we have automatically

$$\forall s > c_6 \quad m_{\mathcal{G}}(\{|\chi| > s\}) \leq c_4 \exp(-(c_5 s^2(\log d/d))).$$

Setting $F = \chi(d/\log d)^{-1/2}$, we find

$$\forall s > c_6 \quad m_{\mathcal{G}}(\{|F| > s\}) \leq c_4 \exp(-(c_5 s^2)),$$

from which it is easy to deduce that, for some c_7 , we have $\|F\|_{\psi_2} \leq c_7$, or equivalently

$$\|\chi\|_{\psi_2} \leq c_7 \sqrt{d/\log d},$$

i.e. the case $G = \mathcal{G}$ shows that the growth when $d \rightarrow \infty$ of the constant in (5.12) is optimal.

When the representations π_n are defined on a single compact group G (so that $G_n = G$ for all n), the next result was proved in [9] for G a Lie group and in [22] for G a profinite group.

Corollary 5.10. *If the groups (G_n) appearing in Theorem 0.1 are finite (or amenable as discrete groups) then the equivalent properties in Theorem 0.1 can hold only if the dimensions $d_n = \dim(\pi_n)$ remain bounded.*

Remark 5.11. The proof of the bound $d + 1!$ in [10] uses the classification of finite simple groups. However, all that is needed for the last Corollary is a bound of the index that is $o(d^2)$. Such a bound, a much easier one, of the order $d^{c(d/\log d)^2} = \exp(cd^2/\log d)$ is known. It is due to Blichfeldt, as indicated in [1, p. 103] and [14, p. 177]. Blichfeldt's bound improved previous ones due to himself, then Bieberbach and Frobenius [17]. See [12, §36] for more on the subject. See [5] for a discussion of Jordan's ideas, and [6] for more recent related results.

Let $f_s(d)$ be the best possible $f(d)$ if one restricts to *solvable* finite subgroups $G \subset U(d)$. In [14, p. 218] L. Dornhoff proves that $f_s(d) = \exp O(d)$ and that this is optimal. Thus, for such groups G , we obtain a better bound:

Corollary 5.12. *There is a numerical constant $C > 0$ such that for any d and any solvable finite subgroup $G \subset U(d)$ we have*

$$\mathbb{E} \sup_{x \in G} |\operatorname{tr}(xg_d)| \leq C\sqrt{d}.$$

Remark 5.13. The preceding bound is essentially optimal since if G is the diagonal (finite Abelian) subgroup of $U(d)$ with entries $= \pm 1$ we have clearly $\mathbb{E} \sup_{x \in G} |\operatorname{tr}(xg_d)| \approx \sqrt{d}$.

6. Sidon sets

Let G be a compact group.

Definition 6.1. A subset $\Lambda \subset \widehat{G}$ is said to be a Sidon set if there is a constant C such that for any family (a_π) with $a_\pi \in M_{d_\pi}$ and $\pi \mapsto a_\pi$ finitely supported, we have

$$(6.1) \quad \sum \operatorname{tr}|a_\pi| \leq C \sup_{t \in G} \left| \sum \operatorname{tr}(a_\pi \pi(t)) \right|.$$

The smallest such C is called the Sidon constant of Λ .

Let $\mathbb{G} = \prod_{\pi \in \widehat{G}} U(d_\pi)$. We say that $\Lambda \subset \widehat{G}$ is randomly Sidon if there is a constant C such that for any family (a_π) as before we have

$$(6.2) \quad \sum \operatorname{tr}|a_\pi| \leq C \int_{\mathbb{G}} \sup_{t \in G} \left| \sum \operatorname{tr}(a_\pi u_\pi \pi(t)) \right| m_{\mathbb{G}}(du).$$

The set Λ is called local Sidon (resp. local randomly Sidon) if (6.1) (resp. (6.2)) only holds for all (a_π) with at most a single non zero term. (Note that these local variants are trivial in the commutative case.)

See [15] and [21] for early results on random Fourier series and lacunary sets. See [19] for a more recent account on Sidon sets.

Obviously Sidon implies randomly Sidon. The converse was announced by Rider in [33] and proved there in the commutative case, but the first (and apparently only) published proof for the non-commutative case appeared only recently in [32]. It follows automatically that local Sidon and local randomly Sidon are also equivalent properties.

Fix $1 < p < \infty$. The set $\Lambda \subset \widehat{G}$ is called a $\Lambda(p)$ -set if there is a constant C such that for any family (a_π) as before the function $F(t) = \sum \operatorname{tr}(a_\pi \pi(t))$ satisfies

$$(6.3) \quad \|F\|_p \leq C\|F\|_1.$$

When $p > 1$ we can replace $\|F\|_1$ by $\|F\|_2$ in this definition. See [3] for more on $\Lambda(p)$ -sets.

Using more recent terminology and recalling (3.1), let us say for short that $\Lambda \subset \widehat{G}$ is subgaussian if there is a constant C such that any F as before satisfies

$$(6.4) \quad \|F\|_{\psi_2} \leq C\|F\|_2.$$

Using (3.2) this can be related to $\Lambda(p)$ -sets. The set Λ is called local $\Lambda(p)$ (resp. local subgaussian) if, for some C , (6.3) (resp. (6.4)) holds for all F of the form $F = \operatorname{tr}(a_\pi \pi(t))$ with $\pi \in \Lambda$.

The adjective “central” is added to any one of the preceding definitions to designate the property obtained by restricting it to families (a_π) formed of scalar multiples of the identity (see [26]).

It was proved by the second author that subgaussian implies Sidon. Since the converse was already known (due to Rudin [36] in the commutative case and to Figà-Talamanca and Rider [16] in the non-commutative one), Sidon and subgaussian are equivalent, and similarly for the local properties. See [27, 28, 32] for more on this.

Note that (i) in Theorem 0.1 means equivalently that the set formed of the coordinates on $\prod_{n \geq 1} G_n$ is a local Sidon set, while (ii) means that it is a central local subgaussian set, and (iii) means that it is a central local randomly Sidon set. Actually using an averaging argument based on the irreducibility of the π 's, it is rather easy to show that a central local randomly Sidon set is local randomly Sidon.

In the non-commutative case, these notions took a serious stepback when it was discovered that for most classical compact groups G there are *no* infinite subsets $\Lambda \subset \widehat{G}$ satisfying them except in the case when the dimensions $(d_\pi)_{\pi \in \Lambda}$ are bounded. More precisely, Cecchini [9] proved that there are no infinite $\Lambda(4)$ -sets in \widehat{G} with unbounded dimensions if G is a compact Lie group. Giulini and Travaglini [18] improving results due to Price and Rider proved that for any compact connected semisimple Lie group G there are no infinite local Λ_p sets for $p > 1$. Related results appear in [26, 34, 35, 13, 20]. Cartwright and McMullen [8] characterized the compact connected groups that admit an infinite local Sidon set, and proved that they contain an infinite Sidon set. Hutchinson [22] proved that there are no infinite central local subgaussian sets with unbounded dimensions for G profinite.

Following the recent paper [4] the second author investigated what remains of Theorem 0.1 when one replaces $t \mapsto \pi_n(t)$ by a matrix valued function $t \mapsto \varphi_n(t)$ on an arbitrary probability space (T, m) satisfying the same moment conditions as $t \mapsto \pi_n(t)$, namely the following:

$$(6.5) \quad \exists C' \forall n \quad \|\varphi_n\|_{L_\infty(M_{d_n})} \leq C'$$

$$(6.6) \quad \forall i, j, k, l \quad \int \varphi_n(i, j) \overline{\varphi_n(k, \ell)} dm = d_n^{-1} \delta_{i, k} \delta_{j, \ell}.$$

In other words, $\{d_n^{1/2} \varphi_n(i, j) \mid 1 \leq i, j \leq d_n\}$ is an orthonormal system for each n . The analogue of the local subgaussian condition is then:

$$(6.7) \quad \exists C'' \forall n \geq 1 \forall a \in M_{d_n} \quad \|\text{tr}(a \varphi_n)\|_{\psi_2} \leq C'' \|\text{tr}(a \varphi_n)\|_2 = C'' (d_n^{-1} \text{tr}|a|^2)^{1/2}.$$

Under these conditions, there is a constant C (depending only on C', C'') such that

$$(6.8) \quad \forall n \geq 1 \forall a \in M_{d_n} \quad \text{tr}|a| \leq C \sup_{t_1, t_2 \in T} |\text{tr}(a \varphi_n(t_1) \varphi_n(t_2))|.$$

This generalizes the implication local subgaussian \Rightarrow local Sidon mentioned above for representations. This is proved in [31, Remark 3.14]. Obviously, in this general setting there is no obstruction preventing φ from having a *finite range*. Nevertheless, if the range of φ is in some sense close to a group it is natural to expect that an analogue of Corollary 5.10 holds. For instance, fix $\varepsilon > 0$ and $\chi \geq 1$. Assume that there is a subgroup $G_n \subset GL(d_n)$, amenable as a discrete group, such that

$$(6.9) \quad \forall n \geq 1 \sup_{u \in G_n} \|u\| \leq \chi \text{ and } \forall t \in T \quad \exists u \in G_n \text{ such that } \|\varphi_n(t) - u\| \leq \varepsilon.$$

Then, here is one possible generalization of Corollary 5.10:

Corollary 6.2. *Assume (6.5) (6.8) and (6.9). If $\varepsilon < (C(C' + \chi))^{-1}$, then $\sup_n d_n < \infty$.*

Proof. Let $\delta = C^{-1} - (C' + \chi)\varepsilon$. By our assumption $\delta > 0$. We first claim that

$$\forall n \geq 1 \quad \forall a \in M_{d_n} \quad \delta \operatorname{tr}|a| \leq \sup_{u \in G_n} |\operatorname{tr}(ua)|.$$

Indeed, by (6.5) and (6.9) for any $t_1, t_2 \in T$

$$|\operatorname{tr}(a\varphi_n(t_1)\varphi_n(t_2))| \leq |\operatorname{tr}(a(\varphi_n(t_1) - u_1)\varphi_n(t_2))| + |\operatorname{tr}(au_1(\varphi_n(t_2) - u_2))| + |\operatorname{tr}(au_1u_2)|$$

and hence

$$|\operatorname{tr}(a\varphi_n(t_1)\varphi_n(t_2))| \leq (C' + \chi)\varepsilon \operatorname{tr}|a| + \sup_{u_1, u_2 \in G_n} |\operatorname{tr}(u_1u_2a)| = (C' + \chi)\varepsilon \operatorname{tr}|a| + \sup_{u \in G_n} |\operatorname{tr}(ua)|.$$

From this and (6.8) the claim is immediate. By a well known averaging argument, since G_n is amenable, there is $v \in GL(d_n)$ with $\|v\|\|v^{-1}\| \leq \chi^2$ such that $G'_n = vG_nv^{-1} \subset U(d_n)$. Then our claim implies

$$\delta \operatorname{tr}|v^{-1}av| \leq \sup_{u \in G'_n} |\operatorname{tr}(ua)|,$$

and hence $\delta \chi^{-2} \operatorname{tr}|a| \leq \delta \operatorname{tr}|v^{-1}av| \leq \sup_{u \in G'_n} |\operatorname{tr}(ua)|$. Thus we conclude by Corollary 5.10 and Remark 0.4. \square

Remark 6.3. In a paper in preparation we plan to give a characterization of the sequences of irreducible unitary representations $\pi_n : G_n \rightarrow U(d_n)$ for which Theorem 0.1 holds. We will give a structural description of the irreducible compact subgroups of $U(d)$ for which the inclusion $\pi : G \rightarrow U(d)$ has a bounded constant $C_2(\pi)$ (with the notation in Lemma 5.1).

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